TRANSIENT TEMPERATURE FIELD IN A FLAT PLATE WITH INTERNAL COMPOUNDING OF HEAT CONDUCTION AND RADIATION

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UDC 536.3

The heating of a flat plate is analyzed, this plate being made of a material with selective radiation absorption characteristics, when the active external convective and radiative fluxes vary with time.

The heating rate of materials with selective radiation absorption characteristics depends on the emission spectrum of the source and on the absorption spectrum of the material. Radiant energy is transmitted with attendant partly total reflection and partly refraction of rays at the intermedia boundaries around the receiver body. As the power of the external radiation source increases and as the body temperature rises, these phenomena play a more important role in the heat propagation inside the body [1].

The temperature field in a flat plate, taking into account these phenomena, was first analyzed in [2] in connection with the heating rate of slabs of optical glass. It was a basic deficiency of that analysis to consider the intensity of the external heat source constant and equal to the intensity of an ideal black-body emitter. The volume of calculations required to account for the internal absorption of radiant energy was so staggering that even with the aid of modern computers only the simplest problems could be solved. An approximate method of solution based on the concept of effective emissivity of a surface [3] is applicable only when the temperature gradients are small.

The method of solution considered here will be based on the hypothesis of small reflectivity and on the concept of characteristic (optically thick and optically thin) radiation absorption layers, which does not essentially restrict the applicability of the solution to a certain class of problems only. The plate material may be any dielectric with a low mean reflectivity.

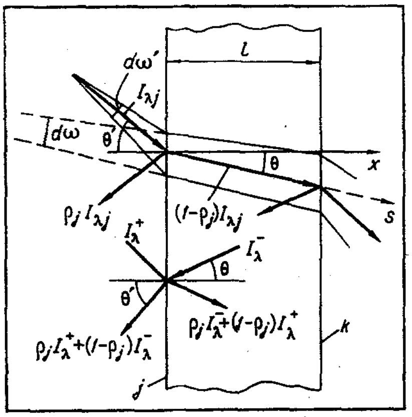

Fig. 1. Schematic diagram depicting the transmission of rays through a flat plate.

Fundamental Equations. The absorption of radiant energy bands $I_\lambda^+ d\omega$ and $I_\lambda^- d\omega$ in an infinitely large plate depends on the optical thickness s in the direction of wave travel, at an angle θ to the plate axis x (Fig. 1):

$$\tau_\lambda(x) = \int_0^s a_\lambda(T)\cos\theta ds = \int_0^x a_\lambda(T)\,dx$$

and, under conditions of local thermodynamic equilibrium, follows the equation

$$\mu\frac{\partial I_\lambda^\pm}{\partial\bar{\tau}} + I_\lambda^\pm(\bar{\tau}, \pm\mu) = \frac{n^2}{\pi}I_{\lambda 0}, \tag{1}$$

with the upper sign (±) for I_λ corresponding to $\bar{\tau} = \tau$ and the lower sign (–) corresponding to $\tau = \tau_0 - \tau$, $\tau_0 = \tau(l)$. According to the Refraction Law

$$\sin\theta' = n\sin\theta \tag{2}$$

the boundary conditions inside the body near the $\bar{\tau} = 0$ boundaries become

Zhukovskii Central Institute of Aerohydrodynamics, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 22, No. 5, pp. 899-906, May, 1972. Original article submitted August 17, 1971.

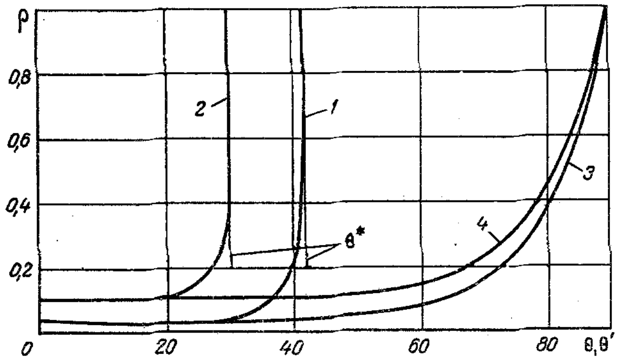

Fig. 2

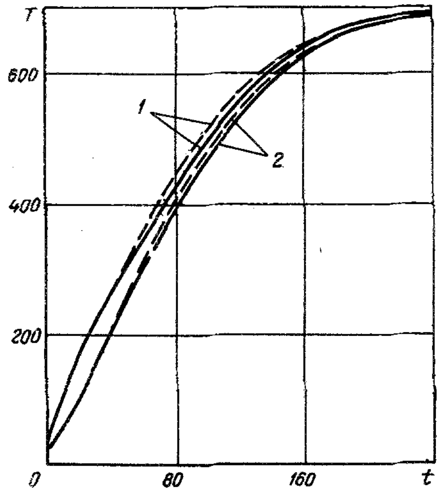

Fig. 3

Fig. 2. Spectral reflectivity of dielectrics, as a function of the incidence and the refraction angle: 1) $\rho(\theta)$ for n = 1.5; 2) $\rho(\theta)$ for n = 2; 3) $\rho(\theta')$ for n = 1.5; 4) $\rho(\theta')$ for n = 2.

Fig. 3. Plate temperature (T, °C) as a function of time (t, sec), a comparison between results of our calculation (solid lines) and the results in [2] (dashed lines) for the temperature at the plate surfaces (1) and at the plate center (2).

$$I_\lambda^{\pm}(0, \pm\mu) = \rho_p(\mu, \mu') I_\lambda^{\mp}(\tau_0, \mp\mu) + [1 - \rho_p(\mu, \mu')] n^2 I_{\lambda p}(\mu', t), \tag{3}$$

with p = j and p = k corresponding respectively to the upper and the lower sign (±) for I_λ.

In the equation of energy transmission

$$\gamma c(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[\Lambda(T)\frac{\partial T}{\partial x} - q_r(T, \bar{T}_p, x, t)\right] \tag{4}$$

there appears a term representing the resultant internal radiation flux

$$q_r = \int\limits_0^\infty \int\limits_{4\pi} (I_\lambda^+ - I_\lambda^-)\,\mu\, d\omega\, d\lambda. \tag{5}$$

The fraction of radiant energy which is absorbed and emitted at the plate surfaces (it can be defined by the criterion $a_\lambda \Delta l > 1$ for a characteristic layer thickness $\Delta l \ll l$) determines, together with the convective fluxes expressed in terms of h, H, or q, the boundary condition for Eq. (4):

$$\delta_p \Lambda(T_p)\frac{\partial T_p}{\partial x} = \int\limits_0^\infty \int\limits_{2\pi} (1 - \rho_p)\left[I_{\lambda p}(\bar{T}_p) - \frac{1}{\pi} I_{\lambda 0}(T_p)\right]\mu' d\omega' d\lambda' + h_p(t)[H_{ep}(t) - H(T_p)] + q_p(t) \quad (p = j, k), \tag{6}$$

where $\delta_p = -1$ for p = j and $\delta_p = 1$ for p = k.

According to Fresnel's law, for intrinsically polarized radiation

$$\rho = \frac{1}{2}(\rho\perp + \rho\parallel),$$
$$\rho\perp = \frac{\sin^2(\theta' - \theta)}{\sin^2(\theta' + \theta)}, \quad \rho\parallel = \frac{\mathrm{tg}^2(\theta' - \theta)}{\mathrm{tg}^2(\theta' + \theta)}, \tag{7}$$

where the indices λ, j, and k of ρ have been omitted. Formulas (2), (3), and (7) are valid for sufficiently smooth surfaces. The phenomena of dispersion or scatter are not considered here.

Solution of the Problem. For a given initial temperature distribution T(x) the problem is defined by relations (1)-(7). An essential difficulty in solving it has to do with the calculation of the q_r-term in (4). We show here a procedure calculating this quantity in the case of zones optically thin and optically thick with respect to the radiation absorption spectrum. After that, the solution of the problem by the explicit finite-difference scheme of approximation presents no basic difficulties and has been carried out with the aid of a computer.

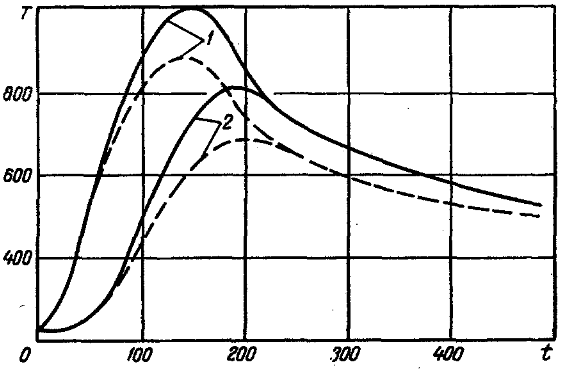

Fig. 4. Temperature of quartz plate surfaces (T, °C) as a function of time (t, sec): temperature of outer surface (1) and of inner surface (2) for a semitranslucent plate (solid lines) and for an opaque ($\varepsilon = 0.91$) plate (dashed lines).

Integrating Eqs. (1) with the conditions (2), inserting the obtained expressions for I_λ^+ and I_λ^- into (5), and then integrating (5) with respect to $\varphi (d\omega = -\mu d\varphi)$, we find

$$q_{\lambda r} = 2\pi n^2 \sum_p \int_0^1 A_{pq}\Big\{(1-\rho_p) I_{\lambda p}\exp\left(-\frac{\bar{\tau}}{\mu}\right) + \rho_p(1-\rho_q) I_{\lambda q}\exp\left(-\frac{\tau_0+\bar{\tau}}{\mu}\right) + \frac{\rho}{\pi\mu}\int_0^{\tau_0} I_{\lambda 0}\left[\exp\left(-\frac{\bar{\tau}_1+\bar{\tau}}{\mu}\right)+\rho_q\exp\left(-\frac{2\tau_0-\bar{\tau}_1+\bar{\tau}}{\mu}\right)\right] \times d\tau_1\Big\}\mu d\mu + 2n^2\left[\int_0^{\tau} I_{\lambda 0}E_2(\tau-\tau_1)\, d\tau_1 - \int_{\tau}^{\tau_0} I_{\lambda 0}E_2(\tau_1-\tau)\, d\tau_1\right] \quad (8)$$

with p, q = j, k and p ≠ q while $\bar{\tau} = \tau$ when p = j, but $\bar{\tau} = \tau_0 - \tau$ and

$$A_{jk} = \left[1-\rho_j\rho_k\exp\left(-\frac{2\tau_0}{\mu}\right)\right]^{-1}, \quad A_{kj} = -A_{jk}, \quad (9)$$

when p = k.

In the last two terms of (8) an additional integration has been performed with respect to μ according to the well known formula

$$E_n(z) = \int_0^1 \mu^{n-2}\exp\left(-\frac{z}{\mu}\right) d\mu. \quad (10)$$

In order to continue integrating with respect to μ, we consider the values of reflectivity for dielectrics in an airless medium. The $\rho = \rho(\theta)$ and $\rho = \rho(\theta')$ characteristics calculated according to (2) and (7) are shown in Fig. 2. The value of $\rho(\theta')$ does not differ much from the value of ρ_n at normal incidence for all angles $\theta' \le 60°$, at which the quantity of emitted energy is relatively high. An examination of the mean hemispherical reflectivity $\bar{\rho}$ of dielectrics – averaged over the incidence angles and over the polarization angles – shows that $\rho_n \ll 1$ when $n \le 2.5$; $1-\bar{\rho}$ is equal to $1-\bar{\rho}_n$ within 5% and is only a weak function of λ [4, 5]. We will, therefore, approximate function $\rho(\theta')$ as follows: for $\theta' < \pi/2$ we replace ρ_j and ρ_k by their mean values $\bar{\rho}_j$ and $\bar{\rho}_k$, considering that $O(\bar{\rho}_j) = O(\bar{\rho}_k) = O(\bar{\rho})$ and $O(\bar{\rho}^2) \ll 1$. With respect to internal angles θ, this corresponds to a piecewise-constant approximation: for $\theta < \theta_*$ we let $\rho_j = \bar{\rho}_j$ and $\rho_k = \bar{\rho}_k$, for $\theta \ge \theta_*$ we let $\bar{\rho}_j = \bar{\rho}_k = 1$. Treating ρ^2 as the small parameter in the problem, we will estimate the terms in (8) with τ_0 within the absorption bands where $\tau_0 > 1$ and τ_0 within the transmission bands where $\tau_0 \ll 1$. Since the integrand functions of the $A_{jk}f$ kind in (8) are sign-definite as far as integration with respect to u inside the braces is concerned and since they do not contain singularities ($\tau_0 \ne 0$), hence expanding (9) into a power series at various values of τ_0 and θ will yield for these terms

$$\int_0^1 A_{jk}(\bar{\rho}, \mu) f(z, \mu)\mu d\mu = A_2\int_0^{\mu_*} f\mu d\mu\,\Big|_{\theta<\theta^*} + A_1\int_{\mu_*}^1 f\mu d\mu\,\Big|_{\theta>\theta_*}, \quad (11)$$

where

$$A_1 \sim 1 + O\left[\rho^2\exp\left(-\frac{2\tau_0}{\mu}\right)\right], \quad A_2 \sim 1 + O\left[\exp\left(-\frac{2\tau_0}{\zeta}\right)\right]$$

and

$$A_1 \sim (1-\bar{\rho}_j\bar{\rho}_k)^{-1} + O\left(\bar{\rho}^2\frac{2\tau_0}{\zeta}\right), \quad A_2 \sim \frac{\mu_*}{2\tau_0} + O(1),$$

where $\tau_0 > 1$ and when $\tau_0 \ll 1$ respectively. Here $\mu_* < \zeta \le 1$ for A_1 and $0 < \zeta \le \mu_*$ for A_2 (the $O(\bar{\rho}^2)$-term has been retained in the expression for A_1, for the purpose of examining certain limitations).

It will be assumed that the radiation sources off the plate surface x = 0 are distributed within an optically thin layer of gas at a high temperature, this layer having a constant thickness b, and that the state of the gas does not depend on the plate temperature. In this case the monochromatic absorptivity of the gas layer at an angle θ' to the x-axis is [6]:

$$\alpha_{\lambda\mu} = 1 - \exp(-\varepsilon_\lambda \varepsilon_\mu) \approx \varepsilon_\lambda \varepsilon_\mu, \tag{12}$$

with $\varepsilon_\mu = 1/\mu'$, $\varepsilon_\lambda = a^r_\lambda b$, and a^r_λ denoting the spectral radiation attenuation factor in a multicomponent mixture of nonreacting gases. The parameter ε_μ, which defines the directional characteristic of radiation $I_{\lambda j}$, may be considered limited ($\rho(\theta')$ increases rapidly at angles $\theta > 60°$, and it suffices to consider only angles smaller than those and to assume, accordingly, that $\varepsilon_\mu < 2$). Thus, according to (12), we have for the $I_{\lambda j}$-terms in (6) and (8), at wavelengths within the absorption (transmission) band of the material,

$$\int_{\Delta_i} I_{\lambda_j} d\lambda' = \frac{1}{\pi} \int_{\Delta_i} \alpha_{\lambda\mu} I_{\lambda 0} d\lambda' = \frac{\varepsilon_\mu}{\pi} \int_{\Delta_i} \varepsilon_\lambda I_{\lambda 0}(\bar{T}_j)\, d\lambda'. \tag{13}$$

For a radiating gas, function ε_λ depends strongly on λ, becoming zero within certain wavelength bands, also generally on the gas temperature as well as pressure: calculating this function makes the numerical solution of the problem more complicated.

On the basis of (10), (11), and (13), all terms in (6) and (8) referred to the wavelength band $d\lambda$ are, after integration with respect to μ, expressed as integrals

$$\int E_2(z)\, I_{\lambda 0}(\tau_1)\, d\tau_1 \text{ and } I_{\lambda 0}(\bar{T}_p) \int E_3(z)\, d\tau_1, \tag{14}$$

where argument z assumes the values τ, $\tau_1 + \tau$, $2\tau_0 - \tau_1 + \tau$, $2\tau_0 + \tau - \tau_1$, $2\tau_0 + \tau_1 - \tau$, $\tau - \tau_1$, $\tau_0 - \tau$, and $2\tau_0 - \tau$. A further transformation with respect to τ_1 and λ is continued within the transmission band Δ_i by means of the asymptotic representation of E(z) near z = 0 and within the absorption band Δ_i by representing the argument of $I_{\lambda 0}(\tau_1)$ as $\tau_1 \equiv z_0 + z$, followed by an expansion of $I_{\lambda 0}(z_0 + z)$ in (14) into a power series near z_0 for small z; such a transformation is shown in, for instance, [5].

The transformation of all terms in (8) for the transmission band with $a_i = a_i(\tau)$ and for the absorption band with a_i = const will yield for those terms

$$\frac{\partial q_r}{\partial x} = 4a_i(\tau)\left[\Phi_i - \sum_p C_{pq} \int_{\Delta_i} \varepsilon_{\lambda p} I_{\lambda 0}(\bar{T}_p)\, d\lambda' - \frac{\mu_*}{\tau_0(a_i)} \int_0^l a_i \Phi_i dx\right],$$

$$\frac{\partial q_r}{\partial x} = -\frac{4}{3a_i} \cdot \frac{\partial^2 \Phi_i}{\partial x^2} + \sum_p \left[F_{ip}\Phi_i - D_{ip} \int_{\Delta_i} \varepsilon_{\lambda p} I_{\lambda 0}(\bar{T}_p)\, d\lambda'\right],$$

$$C_{pq} = \frac{(1-\bar{\rho}_p)(1+\bar{\rho}_q)}{2n(1-\bar{\rho}_p\bar{\rho}_q)} \int_0^1 \frac{\varepsilon_{\mu p}\mu' d\mu'}{\sqrt{n^2\mu_*^2 + \mu'^2}} \quad (p,\ q = j,\ k;\ p \neq q), \tag{15}$$

$$D_{ip} = 2a_i(1-\bar{\rho}_p) \int_0^1 \exp\left(-\frac{n\bar{\tau}_{ip}}{\sqrt{n^2\mu_*^2 + \mu'^2}}\right) \frac{\varepsilon_{\mu p}\mu' d\mu'}{\sqrt{n^2\mu_*^2 + \mu'^2}} \quad (p = j,\ k),$$

$$F_{ip} = 3a_i(1-\bar{\rho}_p)\left[\exp\left(-\frac{3}{2}\bar{\tau}_{ip}\right) - \mu_* \exp\left(-\frac{3}{2} \cdot \frac{\bar{\tau}_{ip}}{\mu_*}\right)\right] \quad (p = j,\ k),$$

$$\Phi_i(n\lambda T) = [\eta(n\lambda_i T) - \eta(n\lambda_{i-1} T)]\, n^2\sigma T^4,$$

where function η has been tabulated in [5]:

$$\eta(n\lambda T) = \frac{1}{n^3\sigma T^5} \int_0^{n\lambda T} I_{\lambda 0}(z)\, dz.$$

For the thermal flux at the surface we find from (6)

$$\delta_p \Lambda_p(T_p) \frac{\partial T_p}{\partial x} = (1-\bar{\rho}_p)\left[2\int_0^1 \varepsilon_{\mu p}\mu' d\mu' \int_{\Delta_i} \varepsilon_{\lambda p} I_{\lambda 0}(\bar{T}_p)\, d\lambda' - \Phi_i(\lambda' T_p)\right] + h_p(t)\,[H_{ep}(t) - H(T_p)] + q_p(t) \quad (p = j,\ k). \tag{16}$$

Formulas (15) and (16) can be used in the derivation of respective formulas for any number of Δ_i bands.

A computer program, on the BESM-3M, has been set up for calculating the temperature of a plate heated from the x = 0 surface by a given radiative flux at a gas temperature $\bar{T}_j(t)$ and by a convective flux, the latter defined as a function of time in terms of enthalpy, or by a flux $q_j(t)$. At the $x = l$ surface is given a convective flux. The quantities c, Λ, and a for the transmission band are functions of the temperature.

Examples. In the first example a 6 mm thick plate of silicate glass is heated from ideal black-body sources radiating at a constant temperature $\bar{T}_j = \bar{T}_k = 707°C$. This simple example is interesting on account of the numerical solution obtained [2] without the asymptotic representations for q_r, which we have used, and without taking into consideration the smallness of parameter ρ. That theoretical approach has been supported also by experiment. The transmission band for these calculations was assumed 1.0-2.75 μ with an attenuation factor $a = a(T)$ and the absorption band was assumed 2.75-4.0 μ with $a = 500\ m^{-1}$. Long-wave radiation (>4.0 μ) was absorbed at the plate surfaces.

The comparison of calculated results shown in Fig. 3 establishes a close agreement between both solutions, within approximately 5%.

In the second example a quartz plate is heated from the x = 0 surface by a thermal flux q(t): $q = 1.5 \cdot 10^{-5} \sin(\pi t/200)$ during t = 200 sec and q = 0 after 200 sec. The other initial data are: n = 1.5, l = 12 mm, T(x) = 300°K at t = 0, the transmission band 0-3.5 μ with $a = 0.18\ m^{-1}$, the absorption band 3.5-4.3 μ with $a = 150\ m^{-1}$, long-wave radiation (>4.3 μ) absorbed at the surface [7], $\gamma = 2220\ kg/m^3$, and the c(T), Λ(T) characteristics taken from [8].

The result of calculations is shown in Fig. 4 in terms of plate surface temperature as a function of time. For comparison, also the temperature calculations for an opaque plate ($q_r \equiv 0$ in (4) and ε = 0.91) are shown here.

This comparison indicates that bulk radiation of thermal energy in quartz affects the temperature field, beginning approximately at a plate temperature of 300°C and then shifting the maximum at higher temperatures, in consequence of a reduced effective emissivity of the plate behaving as a semitranslucent body.

In silicate glasses this effect is weaker, because of their narrower transmission band and the lower allowable temperature limit to which a plate may be heated.

NOTATION

λ	is the wavelength;
l	is the plate thickness;
x	is the coordinate normal to the plane of the plate;
a	is the attenuation factor;
θ', θ	are the polar incidence and refraction angle respectively;
μ = cos θ;	
n	is the refractive index;
τ	is the optical thickness;
T	is the temperature;
s	is the coordinate along the ray;
I_λ^+, I_λ^-	are the spectral intensity of radiation in the forward direction s and in the reverse direction respectively;
ω	is the solid angle;
ρ	is the reflectivity;
t	is the time;
$\bar{T}$	is the characteristic temperature of external radiation source;
γ	is the density of the material;
c	is the specific heat of the material;
Λ	is the thermal conductivity of the material;
q	is the given external thermal flux at the plate surface;
q_r	is the resultant radiation flux;
h	is the heat-transfer coefficient in terms of enthalpy;
$I_{\lambda 0}$	is the spectral hemispherical radiation intensity of an ideal black body;

H_e	is the enthalpy of recovery;
H	is the enthalpy at plate surface temperature;
φ	is the meridional angle;
$\bar{\rho}$	is the mean hemispherical reflectivity;
ρ_n	is the reflectivity at a normal angle of incidence;
ε	is the emissivity;
σ	is the Stefan–Boltzmann constant;

$\tau_{ip} = a_i x$ for $p = j$;
$\tau_{ip} = a_i (l - x)$ for $p = k$.

Subscripts

λ	refers to monochromatic energy;
$(\ldots)'$	refers to outside the body;
j, k	refer to plate surfaces x = 0 and x = 1 respectively;
$(\ldots)_*$	refers to critical refraction angle;
i	refers to i-th wavelength band $\Delta_i = \lambda_i - \lambda_{i-1}$.

LITERATURE CITED

1. B. A. Khrustalev, Inzh. Fiz. Zh., 18, No. 4 (1970).
2. R. Gordon, J. Amer. Ceram. Soc., 41, No. 6 (1958).
3. V. F. Vorob'ev, Uchenye Zapiski Tsentr. Aérogidrodinam. Inst., 1, No. 5 (1970).
4. G. A. Blokh, Principles of Radiative Heat Transfer [in Russian], Izd. GÉI, Moscow–Leningrad (1962).
5. E. M. Sparrow and R. D. Cess, Radiation Heat Transfer, Brooks–Cole Publ. Corp., Belmont Calif. (1969).
6. Turbulent Flow and Heat Transfer [Russian translation], Izd. Inostr. Lit., Moscow (1963).
7. R. Smith, F. John, and R. Chesmer, Detection and Measurement of Infrared Radiation [Russian translation], Izd. Inostr. Lit., Moscow (1959).
8. Handbook of Chemistry [in Russian], Khimiya, Moscow–Leningrad, Vol. 5 (1966).